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Phase Diagrams of Lattice Systems with Residual Entropy. II. Low Temperature Expansion

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For some lattice systems with an infinite number of ground states, it is shown that the pressure and the coexistence surfaces of several phases admit asymptotic expansions around $T=0$. In particular, it follows that the coexistence surfaces are differentiable at $T=0$, and at low temperatures the stable states are those with maximal residual entropy. The results are applied to construct the phase diagrams for several spin-1 models.

KEY WORDS: Phase diagram; first-order phase transition; residual entropy; pressure; coexistence surface; asymptotic expansion.

1. INTRODUCTION

This paper is the continuation of our earlier work[1] in which we studied the phase diagram of lattice systems with residual entropy. In these systems, the number of ground state configurations (gsc) in finite volumes increases exponentially with the volume. Our main idea was to introduce a partition of the set of gsc into equivalence classes and to replace, if possible, the finite number of periodic gsc for the Pigorov–Sinai (PS) theory[2,3] by the finite number of periodic classes. Under certain conditions which we recall in Section 2, we could extend the PS theory for this case. In the present paper, we adopt Slawny's method[4] to describe the low-temperature phase diagram for these models.

Given an r-parameter family of Hamiltonians $H(\boldsymbol{\lambda})$, $\boldsymbol{\lambda} \in \mathbb{R}^r$, the phase diagram at zero temperature is a partition $\{A_I(T=0)\}$ of the parameter

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space $\mathbb{R}^r$ into domains where the gsc are the same: the subscript I labels the set of configurations which are gsc in the given domain. One is then interested to know whether this partition will deform continuously as the temperature $T = 1/\beta$ is raised, to give domains $\{A_I(T)\}$ of pure phases of $H(\boldsymbol{\lambda})$ which are small perturbations of the corresponding gsc. To answer this question, one singles out a point element $A_{I^0}(0) = \{\boldsymbol{\lambda}_0\}$ of the zero-temperature partition. If $H(\boldsymbol{\lambda}_0)$ has $d = r + 1$ gsc and $\{A_I(0)\}$ in a neighborhood of $\boldsymbol{\lambda}_0$ is homeomorphic to the natural partition of the boundary of the positive octant in $\mathbb{R}^d$, then the PS theory (or our extention) proves the continuity of the phase diagram at $T = 0$. Without restricting the generality, one can choose $\boldsymbol{\lambda}_0 = 0$. For a fixed $\boldsymbol{\lambda}^*$ in a neighborhood of $\boldsymbol{\lambda}_0$, one would like to know which of the gsc of $H(\boldsymbol{\lambda}^*)$ will be stable with respect to thermal perturbations. Clearly, the knowledge of $A_I(T)$ for small T permits us to identify the stable gsc of $H(\boldsymbol{\lambda}^*)$. A method to determine the low-temperature form of $A_I(T)$ is proposed by Slawny.[4] Consider $A_{I^0}(T) = \{\boldsymbol{\lambda}(T)\}$, i.e., a single point for any T, where $\boldsymbol{\lambda}(0) = \boldsymbol{\lambda}_0$. In the case when $H(\boldsymbol{\lambda}_0)$ has a finite number $(d = r + 1)$ of periodic gsc, Slawny shows that the function $\boldsymbol{\mu}(\beta) = \beta \boldsymbol{\lambda}(1/\beta)$ has an asymptotic expansion of the form $\sum_{n \geqslant 1} \boldsymbol{\mu}_n \exp[-\beta E_n]$, where $0 < E_1 < E_2 < \cdots$, and gives prescriptions to calculate the coefficients of this series. This method can easily be adapted to describe the phase diagram in the neighborhood of a general $\boldsymbol{\lambda}_1 \in A_I(0)$ (see Section 4).

Successful extensions of the PS theory and Slawny's method were obtained in recent works by Bricmont and Slawny,[5,6] where periodic gsc are replaced by restricted ensembles.[7] Their method, however, cannot be applied to models with residual entropy, our actual concern.

The modification to add to Slawny's description in order to include models with residual entropy is simple to summarize, once we know that the PS theory applies. The limit of $\boldsymbol{\mu}(\beta)$ as $\beta \to \infty$ will be nonvanishing, i.e., the asymptotic expansion of $\boldsymbol{\mu}(\beta)$ starts with a nonzero constant $\boldsymbol{\mu}_0$ depending linearly on the differences in the residual entropies of the different classes. This has as a consequence that the coexistence curve $\boldsymbol{\lambda}(T)$ is differentiable at $T = 0$ with a nonvanishing derivative, in contrast with the vanishing derivative of $\boldsymbol{\lambda}(T)$ at $T = 0$ in the problems considered by Slawny and by Bricmont and Slawny. This permits us to write $A_I(T) \approx A_I(0) + T\boldsymbol{\mu}_0$, and to determine the domains of weak stability (as defined by Dobrushin and Shlosman[8]) of the different classes. In particular, we obtain that for fixed $\boldsymbol{\lambda}$, the stable classes of $H(\boldsymbol{\lambda})$ have maximal residual entropy, reproducing thereby a result by Aizenman and Lieb.[9]

In the next section, we describe the precise conditions under which our treatment is applicable. In Section 3, we discuss the low-temperature expansion appropriate in our case and announce a theorem on the asymptotic expansion of the pressure in the thermodynamic limit. In Section 4, we formulate a theorem on the asymptotic expansion of the coexistence curve $\lambda(T)$. Section 5 contains the proofs of the two theorems and Section 6 the conclusions that the results imply for the phase diagram. We present several illustrations on spin-1 models in the last section.

2. DESCRIPTION OF THE SYSTEM

Let $\mathbb{L} = \mathbb{Z}^\nu$ be a ν-dimensional simple cubic lattice. With each site $i \in \mathbb{L}$ is associated a finite set $\Omega_i \cong \Omega_0$, where $x(i) \in \Omega_i$ is the "value of the spin" at i. The configuration space is $\Omega = \Omega_0^{\mathbb{L}}$. For any $x \in \Omega$ and any $A \subset \mathbb{L}$, $x_A = \{x(i) \mid i \in A\} \in \Omega_0^A = \Omega_A$.

The system is described by the formal Hamiltonian

$$H_0(x) = \sum_B \phi_B^{(0)}(x), \qquad \phi_B^{(0)}(x) = \phi_B^{(0)}(x_B), \qquad (B \subset \mathbb{L}) \tag{2.1}$$

We assume that the interaction is of finite range and periodic (i.e., invariant under a subgroup $\hat{\mathbb{L}}$ of $\mathbb{L}$ with finite index). Without loss of generality we shall suppose that $\min\{\phi_B^{(0)}(x) \mid x \in \Omega\} = 0$. The main assumptions on the interaction are the following:

(i) $\phi^{(0)}$ is an *m-potential*, i.e., the following set $G[H_0]$ is nonempty and defines the ground-state configurations:

$$G[H_0] = \{s \in \Omega \mid \phi_B^{(0)}(s) = 0, \quad \forall B \subset \mathbb{L}\}$$

Thus, $s \in G[H_0]$ minimizes each $\phi_B^{(0)}$.

(ii) $\phi^{(0)}$ satisfies the *factorization condition*: there is a partition $\mathbb{L} = \bigcup_\alpha A_\alpha$ of $\mathbb{L}$ into rectangular cells $A_\alpha = A_0 + t_\alpha$ with $t_\alpha \in \hat{\mathbb{L}}$, and there is a partition of Ω_{A_0},

$$\Omega_{A_0} = \bigcup_{p=1}^{d+1} \Omega_{A_0}^{(p)} \tag{2.2}$$

such that

$$G[H_0] = \bigcup_{p=1}^{d} G_p \qquad \text{with} \quad G_p = \bigoplus_\alpha \Omega_{A_\alpha}^{(p)} \tag{2.3}$$

Here, $\Omega_{A_\alpha}^{(p)} \cong \Omega_{A_0}^{(p)}$. Notice that any class G_p is invariant under a subgroup $\mathbb{L}_p$ of $\hat{\mathbb{L}}$ which contains at least all the t_α.

The *residual entropy* of the class G_p is

$$\sigma_p = \frac{1}{|A_0|} \log |\Omega_{A_0}^{(p)}|, \qquad p = 1, \dots, d \tag{2.4}$$

We consider H_0 to be embedded into a $(d-1)$-parameter family of Hamiltonians with factorizable m-potentials:

$$H = H_0 + \boldsymbol{\lambda} \cdot \mathbf{H}' \tag{2.5}$$

where

$$\boldsymbol{\lambda} = (\lambda^{(j)}) \in \mathbb{R}^{d-1} \tag{2.6}$$

and

$$\mathbf{H}' = (H^{(j)}) = \sum_B (\phi_B^{(j)}) = \sum_B \boldsymbol{\phi}'_B$$
$$\boldsymbol{\phi}'_B\colon \quad \Omega \to \mathbb{R}^{d-1} \tag{2.7}$$

We suppose that $\mathbf{H}'$ completely splits the degeneracy of the classes, but preserves it within each class:

(a) $\phi_B^{(j)}(s_1) = \phi_B^{(j)}(s_2)$ for any j, p and any $s_1, s_2 \in G_p$.

(b) Let $e_p^{(j)}$ denote the energy density of the configurations in G_p with respect to $H^{(j)}$,

$$e_p^{(j)} = e^{(j)}(s) = \frac{1}{|A_0|} \sum_{i \in A_0} \sum_{B \ni i} \phi_B^{(j)}(s)/|B|, \qquad s \in G_p \tag{2.8}$$

Then the matrix $\mathbf{E}$ given by

$$E_{ij} = e_{i+1}^{(j)} - e_1^{(j)} \tag{2.9}$$

is invertible, i.e., the vectors $\mathbf{e}_2 - \mathbf{e}_1, \ldots, \mathbf{e}_d - \mathbf{e}_1$, where

$$\mathbf{e}_p = (e_p^{(j)}) \in \mathbb{R}^{d-1} \tag{2.10}$$

are linearly independent.

Let us remark that $|\det \mathbf{E}|$ is unchanged if $\mathbf{e}_1$ is replaced by $\mathbf{e}_k$ in the definition of $\mathbf{E}$, so that the condition (b) is symmetric with respect to the ground states.

Then, for any sufficiently small $\boldsymbol{\lambda}$, there exists $I \subset \{1, \ldots, d\}$ such that

$$G[H] = \bigcup_{p \in I} G_p \tag{2.11}$$

where $G[H]$ is the set of configurations minimizing $\phi_B = \phi_B^{(0)} + \boldsymbol{\lambda} \cdot \boldsymbol{\phi}'_B$ for each B. Therefore H is also defined by an m-potential, satisfying the factorization condition.

In the following, we shall use the notation

$$\boldsymbol{\mu} = \beta \boldsymbol{\lambda} = (\mu^{(j)}) \qquad (\beta = T^{-1} = \text{inverse temperature}) \tag{2.12}$$

3. LOW-TEMPERATURE EXPANSIONS

To avoid surface contributions in the LT expansions in finite volumes, we will work with some sets $\Omega(\Lambda \mid p)$ of configurations such that $x \in \Omega(\Lambda \mid p)$ is both periodic and percolating for the class G_p. More precisely, for any configuration $x \in \Omega$, consider the union of wrong cells with respect to G_p,

$$\begin{aligned} \mathscr{A}_p(x) &= \bigcup_{\alpha} \{A_\alpha \mid x_{A_\alpha} \notin \Omega^{(p)}_{A_\alpha}\} \\ S_p(x) &= \bigcup_{\beta} \{A_\beta \mid \operatorname{dist}(A_\beta, \mathscr{A}_p(x)) \leqslant R\} \end{aligned} \tag{3.1}$$

where R denotes the range of the interaction. Let Λ be a finite rectangular subset of $\mathbb{L}$, union of cells A_α. With any $x_\Lambda \in \Omega_\Lambda$ we associate the periodic configuration $x \in \Omega$ defined by the translates of x_Λ. By definition, $x \in \Omega(\Lambda \mid p)$ if every component of $S_p(x)$ is finite [component of $S_p(x) =$ maximal connected subset of $S_p(x)$].

Any $x \in \Omega(\Lambda \mid p)$ can be represented on the torus T_Λ. Consider the partition $\mathbb{L} = \bigcup_i \Lambda_i$ into translates of Λ,

$$\Lambda_i = \Lambda + \tau_\Lambda(i), \qquad \mathbb{L}_\Lambda = \{\tau_\Lambda(i)\} \subset \hat{\mathbb{L}} \tag{3.2}$$

Then $S_p(x)$ will be represented by $S_p(x) \bmod \Lambda$. The $\mathbb{L}_\Lambda$-inequivalent components of $S_p(x)$ remain disconnected on T_Λ and the representation of any component M on T_Λ is faithful in the following sense:

$$|M \bmod \Lambda| = |M|$$

$$\begin{aligned} &|\{\text{nearest neighbor pairs of } M \bmod \Lambda\}| \\ &\quad = |\{\text{nearest neighbor pairs of } M\}| \end{aligned}$$

Observe that $\Omega(\Lambda \mid p)$ is $\mathbb{L}_p$-invariant.

The periodic Hamiltonian is the function on $\Omega(\Lambda \mid p)$ given by

$$H^{\text{per}}_\Lambda = \sum_{i \in \Lambda} \sum_{B \ni i} \frac{1}{|B|} (\beta \phi^{(0)}_B + \boldsymbol{\mu} \cdot \boldsymbol{\phi}'_B) \tag{3.3}$$

and the thermodynamic *partition function* is

$$Z^{\text{th}}(\Lambda \mid p) = \sum_{x \in \Omega(\Lambda \mid p)} \exp[-H^{\text{per}}_\Lambda(x)] = [\exp(-|\Lambda| \mathbf{e}_p \cdot \boldsymbol{\mu})]\, Z(\Lambda \mid p) \tag{3.4}$$

$$Z(\Lambda \mid p) = \sum_{x \in \Omega(\Lambda \mid p)} \exp[-H^{\text{per}}_\Lambda(x \mid s)], \qquad s \in G_p \tag{3.5}$$

Here and in the sequel $x|s$ in the argument of a Hamiltonian means that $\phi_B^{(j)}(x)$ has to be replaced by $\phi_B^{(j)}(x)-\phi_B^{(j)}(s)$.

Furthermore, the factorization condition implies

$$\lim_{\beta\to\infty}\frac{1}{|\Lambda|}\log Z(\Lambda\,|\,p)=\sigma_p \tag{3.6}$$

The corresponding *pressures* are

$$P_\Lambda^{\text{th}}(\beta,\boldsymbol{\mu}\,|\,p)=\frac{1}{|\Lambda|}\log Z^{\text{th}}(\Lambda\,|\,p) \tag{3.7}$$

$$P_\Lambda(\beta,\boldsymbol{\mu}\,|\,p)=\frac{1}{|\Lambda|}\log Z(\Lambda\,|\,p) \tag{3.8}$$

i.e.,

$$P_\Lambda^{\text{th}}(\beta,\boldsymbol{\mu}\,|\,p)=-\mathbf{e}_p\cdot\boldsymbol{\mu}+P_\Lambda(\beta,\boldsymbol{\mu}\,|\,p) \tag{3.9}$$

In the infinite-volume limit the thermodynamic pressure P^{th}, defined by the limit $\Lambda\to\mathbb{L}$ of (3.7), is independent of the boundary condition. It is related to P, defined by the limit $\Lambda\to\mathbb{L}$ of (3.8), by the equation

$$P^{\text{th}}(\beta,\boldsymbol{\mu})=-\mathbf{e}_p\cdot\boldsymbol{\mu}+P(\beta,\boldsymbol{\mu}\,|\,p) \tag{3.10}$$

We shall now introduce the LT expansion of the partition function and of the pressure in terms of excitations with respect to the class G_p.

An *elementary excitation* $\xi=[x]$ with respect to G_p is a class of configurations in Ω with the same connected, finite support $S(\xi)=S_p(x)$, and the same configuration on the support. For $t\in\mathbb{L}$, $T_t[x]=[T_tx]$ is the translate of $[x]$ where $(T_tx)(i)=x(i-t)$. Let X^p denote the family of all elementary excitations with respect to G_p. A multiplicity function θ is a mapping $\xi\to\theta(\xi)$ from X^p into the nonnegative integers such that $\theta(\xi)=0$ for almost all ξ. Let $[X^p]$ denote the family of all multiplicity functions on X^p. For any $\mathbb{L}'\subset\mathbb{L}$, θ_1 and $\theta_2\in[X^p]$ are called $\mathbb{L}'$-equivalent if there is some $t\in\mathbb{L}'$ such that $\theta_2(\xi)=\theta_1(T_t\xi)$ for all $\xi\in X^p$. Below, $\sum^{\mathbb{L}'}$ will mean summation over $\mathbb{L}'$-inequivalent multiplicity functions.

Using the factorization condition, we have

$$Z(\Lambda\,|\,p)=e^{|\Lambda|\sigma_p}\sum_{\substack{\theta\in[X^p]\\ \theta\leqslant 1}}^{\mathbb{L}_\Lambda}\prod_{\xi\in X^p}e^{-\theta(\xi)[H(\xi)+\sigma_p|S(\xi)|]}\,D_\Lambda(\theta) \tag{3.11}$$

Here

$$H(\xi)=\beta H_0(\xi)+\boldsymbol{\mu}\cdot\mathbf{H}'(\xi)=\beta H_0(x\,|\,s)+\boldsymbol{\mu}\cdot\mathbf{H}'(x\,|\,s) \tag{3.12}$$

if $\xi = [x]$ and $s \in G_p$, and

$$D_\Lambda(\theta) = \begin{cases} 1 & \text{if the representations of } \{S(\xi) \mid \theta(\xi) = 1\} \\ & \text{on } T_\Lambda \text{ are pairwise disconnected and faithful} \\ 0 & \text{otherwise} \end{cases}$$

Let

$$\mathscr{E} = \{H_0(x \mid s) \mid s \in G[H_0], x = s \text{ a.e.}\} \tag{3.13}$$

denote the discrete additive set of excitation energies for H_0. Thus,

$$\begin{gathered} \mathscr{E} = \{E_0, E_1, E_2, ...\} \\ E_0 = 0 < E_1 < E_2 < \cdots \\ E_i + E_j \in \mathscr{E} \text{ for any } i, j \end{gathered} \tag{3.14}$$

Following standard techniques,[10] we obtain the expansion

$$P_\Lambda(\beta, \boldsymbol{\mu} \mid p) = \sigma_p + \sum_{n \geqslant 1} a_n^{(\Lambda)}(\boldsymbol{\mu} \mid p) e^{-\beta E_n} \tag{3.15}$$

$$a_n^{(\Lambda)}(\boldsymbol{\mu} \mid p) = |\mathbb{L}/\mathbb{L}_p|^{-1} \sum_{\substack{\theta \in [X^p] \\ H_0(\theta) = E_n}}^{\mathbb{L}_p} \Phi(\theta)\, C_\Lambda(\theta) \tag{3.16}$$

$$\Phi(\theta) = \prod_{\xi \in X^p} \frac{1}{\theta(\xi)!} \exp\{-\theta(\xi)[\boldsymbol{\mu} \cdot \mathbf{H}'(\xi) + \sigma_p |S(\xi)|]\} \tag{3.17}$$

where

$$H_0(\theta) = \sum_\xi \theta(\xi)\, H_0(\xi)$$

$C_\Lambda(\theta)$ is a combinational factor that we give in the Appendix. For any fixed θ, it agrees with the usual factor,[10] $C_\Lambda(\theta) = C(\theta)$, if Λ is sufficiently large.

We notice that for any finite Λ the expansions (3.15) converge for $\beta > \beta_0(\Lambda, \boldsymbol{\mu}, p)$. The conditions on the interaction imply that excitations with a finite energy have a finite support. It then follows that for Λ sufficiently large

$$a_n^{(\Lambda)}(\boldsymbol{\mu} \mid p) = a_n(\boldsymbol{\mu} \mid p) = |\mathbb{L}/\mathbb{L}_p|^{-1} \sum_{\substack{\theta \in [X^p] \\ H_0(\theta) = E_n}}^{\mathbb{L}_p} \Phi(\theta)\, C(\theta) \tag{3.18}$$

In other words, as $\Lambda \to \mathbb{L}$ the formal series (3.15) converges in the space $\mathbb{D}$ of formal series to

$$\dot{P}(\beta, \boldsymbol{\mu} \mid p) = \sigma_p + \sum_{n \geqslant 1} a_n(\boldsymbol{\mu} \mid p) e^{-\beta E_n} \tag{3.19}$$

However, since the asymptotic expansion of a function is unique,

$$-\mathbf{e}_p \cdot \boldsymbol{\mu} + \dot{P}(\beta, \boldsymbol{\mu} \mid p) \tag{3.20}$$

cannot be the asymptotic expansion of the pressure $P^{\mathrm{th}}(\beta, \boldsymbol{\mu})$ for all p (unless $\boldsymbol{\mu}=0$ and the classes G_p are related by symmetries of the Hamiltonian). In Section 5, we prove the following result.

Theorem 1. Consider the (temperature-dependent) Hamiltonian

$$H(\beta) = H_0 + \beta^{-1}\boldsymbol{\mu}(\beta) \cdot \mathbf{H}'$$

where $\boldsymbol{\mu}(\beta)$ is an arbitrary function which admits an asymptotic expansion around $\beta = \infty$:

$$\boldsymbol{\mu}(\beta) \sim \dot{\boldsymbol{\mu}}(\beta) = \sum_{n \geqslant 0} \boldsymbol{\mu}_n e^{-\beta E_n} \tag{3.21}$$

If for all $\beta > \beta_c$ there exists an equilibrium state of $H(\beta)$ which is a small perturbation of the class G_p, then the thermodynamic pressure $P^{\mathrm{th}}(\beta) = P^{\mathrm{th}}(\beta, \boldsymbol{\mu}(\beta))$ has an asymptotic expansion and

$$P^{\mathrm{th}}(\beta) \sim -\mathbf{e}_p \cdot \dot{\boldsymbol{\mu}}(\beta) + \dot{P}(\beta, \dot{\boldsymbol{\mu}}(\beta) \mid p) \tag{3.22}$$

4. ASYMPTOTIC EXPANSION OF THE PHASE DIAGRAM

In ref. 1 we extended the Pirogov–Sinai theory to m-potentials satisfying the factorization condition. This implies, among others, that there is some curve $\boldsymbol{\lambda}(T)$, $T = \beta^{-1}$, along which all the phases coexist, i.e., for $H = H_0 + \boldsymbol{\lambda}(T) \cdot \mathbf{H}'$ there are d pure phases belonging respectively to $G_1,..., G_d$ if $T < T_c$.

Theorem 2. The function $\boldsymbol{\mu}(\beta) = \beta\boldsymbol{\lambda}(\beta^{-1})$, where $\boldsymbol{\lambda}(T)$ is the coexistence curve of all the d phases, admits an asymptotic expansion of the form (3.21).

This theorem will be proved in the next section. Then, according to Theorem 1, $\dot{\boldsymbol{\mu}}$ satisfies the system of equations

$$(\mathbf{e}_p - \mathbf{e}_1) \cdot \dot{\boldsymbol{\mu}} = \dot{P}(\beta, \dot{\boldsymbol{\mu}} \mid p) - \dot{P}(\beta, \dot{\boldsymbol{\mu}} \mid 1), \qquad p = 2,..., d \tag{4.1}$$

This can be written in the vectorial form $\mathbf{E}\dot{\boldsymbol{\mu}} = \boldsymbol{\Delta}\dot{P}(\beta, \dot{\boldsymbol{\mu}})$ or

$$\dot{\boldsymbol{\mu}} = \mathbf{E}^{-1}\, \boldsymbol{\Delta}\dot{P}(\beta, \dot{\boldsymbol{\mu}}) = \mathbf{f}(\dot{\boldsymbol{\mu}}) \tag{4.2}$$

for $\mathbf{E}$ is invertible [cf. Eq. (2.9)]. Let

$$\mathbf{f}^{[k]} = \mathbf{f} \circ \mathbf{f} \circ \cdots \mathbf{f} \quad (k \text{ times})$$

Proposition 1. Equation (4.2) has a unique solution $\dot{\boldsymbol{\mu}}(\beta) = \sum_{n \geq 0} \boldsymbol{\mu}_n \exp(-\beta E_n)$ in $\mathbb{D}^{d-1}$, given by

$$\boldsymbol{\mu}_n = \mathbf{f}^{[n+1]}(\mathbf{0})_n \tag{4.3}$$

where the rhs is the coefficient of $\exp(-\beta E_n)$ in the series $\mathbf{f}^{[n+1]}$ $(\dot{\boldsymbol{\mu}} = \mathbf{0})$.

Proof. One can check easily that $\mathbb{D}^r$, $r \geq 1$, equipped with the distance

$$d_\beta(\dot{\boldsymbol{\mu}}, \dot{\mathbf{v}}) = e^{-\beta E_n} \qquad \text{if} \quad \min\{k \mid \boldsymbol{\mu}_k \neq \mathbf{v}_k\} = n$$

is a complete (ultra) metric space. Therefore, one has to show that $\mathbf{f}$ is a contraction in $\mathbb{D}^{d-1}$ to conclude that Eq. (4.2) has a unique solution which is the fixed point of the iteration

$$\dot{\boldsymbol{\varphi}}_{k+1} = \mathbf{f}(\dot{\boldsymbol{\varphi}}_k)$$

Now $\dot{P}$ has the form

$$\dot{P}(\beta, \dot{\boldsymbol{\mu}} \mid p) = \sigma_p + \sum_{n \geq 1} \exp(-\beta E_n) \sum_{\theta: H_0(\theta) = E_n}^{\mathbb{L}_p} \mathrm{A}(\theta) \exp[-\dot{\boldsymbol{\mu}} \cdot \mathbf{B}(\theta)] \tag{4.4}$$

where

$$\begin{aligned} &\exp(-a\dot{\mu}^{(j)}) \\ &\quad = [\exp(-a\mu_0^{(j)})] \\ &\quad\quad \times \left\{1 + \sum_{m \geq 1} \left[\sum_{k \geq 1} \frac{a^k}{k!} \sum_{\substack{(\alpha_1, \ldots, \alpha_k) \\ \alpha_i \neq 0 \\ \sum_i E_{\alpha_i} = E_m}} \prod_{i=1}^{k} \mu_{\alpha_i}^{(j)}\right] \exp(-\beta E_m)\right\} \end{aligned} \tag{4.5}$$

Since

$$\dot{\mu}^{(j)} \to C + \exp(-\beta E_n) \exp(-a\dot{\mu}^{(j)})$$

is a contraction in $\mathbb{D}$ for any $n \geq 1$, $\dot{\boldsymbol{\mu}} \to \Delta \dot{P}(\dot{\boldsymbol{\mu}})$ will be a contraction in $\mathbb{D}^{d-1}$, and the linear transformation $\mathbf{E}^{-1}$ does not affect this property. Then, due to the convergence to the fixed point, (4.3) holds true. In (4.3), the number of iterations $n+1$ could be replaced by any $k > n$.

Due to Theorems 1 and 2 and Proposition 1, the solution $\dot{\boldsymbol{\mu}}(\beta)$ thus obtained is the asymptotic expansion of the coexistence curve $\boldsymbol{\mu}(\beta)$. Let us notice that in order to calculate $\boldsymbol{\mu}_n$, one could replace $\dot{P}(\beta, \dot{\boldsymbol{\mu}} \mid p)$ in Eq. (4.1) by $\dot{P}_\Lambda(\beta, \dot{\boldsymbol{\mu}} \mid p)$ [cf. Eq. (3.15)] with $\Lambda \supset \Lambda(n)$. The equations written with P_Λ determine the "coexistence curve $\boldsymbol{\mu}_\Lambda(\beta)$ for a finite system."

Applying the proposition to $n=0$, we obtain

$$\boldsymbol{\mu}_0 = \mathbf{E}^{-1}\,\Delta\sigma \tag{4.6}$$

with

$$(\Delta\sigma)_i = \sigma_{i+1} - \sigma_1, \qquad i = 1,..., d-1$$

The property $\boldsymbol{\mu}(\beta) \sim \dot{\boldsymbol{\mu}}(\beta)$ implies that $\boldsymbol{\mu}(\beta) \to \boldsymbol{\mu}_0$ as $\beta \to \infty$, which means that the coexistence curve $\boldsymbol{\lambda}(T)$ of all the d phases is differentiable at $T=0$ and

$$\boldsymbol{\lambda}(T) = T\boldsymbol{\mu}_0 + o(T) \qquad \text{as} \quad T \to 0$$

Let $I \subset \{1, 2,..., d\}$ and $\Lambda_I(0) \subset \mathbb{R}^{d-1}$ such that for $\boldsymbol{\lambda} \in \Lambda_I(0)$, $G[H_0 + \boldsymbol{\lambda} \cdot \mathbf{H}'] = \bigcup_{p \in I} G_p$, i.e., $\Lambda_I(0)$ is the zero-temperature coexistence domain of $|I|$ phases (see Introduction). For a given $\boldsymbol{\lambda}_1 \in \Lambda_I(0)$, we can prove Proposition 1 and Theorem 2 with $\boldsymbol{\lambda}_1$ replacing $\boldsymbol{\lambda}_0 = \mathbf{0}$: it is sufficient to restrict the phase diagram to different $(|I|-1)$-dimensional subspaces passing through $\boldsymbol{\lambda}_1$. In almost all these subspaces the conditions of Section 2 will be satisfied with $\boldsymbol{\lambda}_0$ replaced by $\boldsymbol{\lambda}'_0$, where $\boldsymbol{\lambda}'_0$ is an $(|I|-1)$-dimensional vector representing $\boldsymbol{\lambda}_1$ in the given subspace. Then the point $\boldsymbol{\lambda}(T) \cong \boldsymbol{\lambda}'(T)$, where $\boldsymbol{\lambda}(0) = \boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}'(0) = \boldsymbol{\lambda}'_0$, for fixed T is the intersection of the $(d-|I|)$-dimensional set $\Lambda_I(T)$ with the subspace in question. We obtain the differentiability at $T=0$ of all the curves $\boldsymbol{\lambda}(T)$ belonging to different intersections. Therefore we conclude that the coexistence surface $\Lambda_I(T)$ is differentiable at $T=0$.

5. LOW-TEMPERATURE PHASE DIAGRAM

To prove Theorems 1 and 2, one has to use the Pirogov–Sinai (PS) theory. For the appropriate generalization of the PS theory and for definitions and notations, we refer to our earlier paper.[1] One introduces the crystal partition function $Z(\Gamma \mid \beta H)$ for the spin system with Hamiltonian (2.5) and $Z(\Gamma \mid F)$ for contour models. Here $\Gamma \in \mathscr{C}_p$ denotes a contour of type $G_p \subset G[H_0]$, and F is a τ-functional on $\mathscr{C}_p$. One then looks for functionals $\{F_1,..., F_d\}$ which solve the equations

$$e^{-|V(\Gamma)|\sigma_p} Z(\Gamma \mid \beta H) = e^{b_p |\mathrm{Int}\,\Gamma|} Z(\Gamma \mid F_p), \qquad \Gamma \in \mathscr{C}_p$$

$$b_p = \mathbf{e}_p \cdot \boldsymbol{\mu} - \sigma_p - \pi(F_p) - \min_q \{\mathbf{e}_q \cdot \boldsymbol{\mu} - \sigma_q - \pi(F_q)\}$$

where $\pi(F_p)$ is the pressure for the contour model on $\mathscr{C}_p$. For β sufficiently large, there exists a unique solution $\{F_q = F_q(\beta, \boldsymbol{\mu})\}$. Furthermore, for all p such that $b_p(\beta, \boldsymbol{\mu}) = 0$, there is a pure phase associated with G_p and

$$P^{\text{th}}(\beta, \boldsymbol{\mu}) = \pi(F_p(\beta, \boldsymbol{\mu})) + \sigma_p - \mathbf{e}_p \cdot \boldsymbol{\mu}$$
$$= \max_q \{\pi(F_q(\beta, \boldsymbol{\mu})) + \sigma_q - \mathbf{e}_q \cdot \boldsymbol{\mu}\} \tag{5.1}$$

In particular, if $\boldsymbol{\mu}(\beta)$ corresponds to the coexistence of the phases G_p and G_q, then

$$(\mathbf{e}_p - \mathbf{e}_q) \cdot \boldsymbol{\mu} = \sigma_p - \sigma_q + \pi(F_p) - \pi(F_q) \tag{5.2}$$

Following Slawny,[4] we introduce a pressure with cutoff E for the spin system,

$$P_E(\beta, \boldsymbol{\mu} | p) = \lim_{\Lambda \to \infty} |\Lambda|^{-1} \log \sum\nolimits^E \exp[-H_\Lambda^{\text{per}}(x | s)]$$

where the sum is restricted to configurations $x \in \Omega(\Lambda \mid p)$ such that for any elementary excitation ξ associated with x, $H_0(\xi) \leqslant E$.

In this case, $P_E(\beta, \boldsymbol{\mu} \mid p)$ can be represented by an expansion where the coefficients $a_n^E(\boldsymbol{\mu} \mid p)$ are calculated from Eq. (3.18) with the additional restriction that $\theta(\xi) = 0$ if $H_0(\xi) > E$. Since the potential is regular, one can show, using standard methods,[10] that the expansion of $P_E(\beta, \boldsymbol{\mu} \mid p)$ is absolutely convergent for $\beta > \beta_0(E)$. Moreover, $a_n^E(\boldsymbol{\mu} \mid p) = a_n(\boldsymbol{\mu} \mid p)$ for all n with $E_n \leqslant E$.

Following Slawny, one can then show that for any E there exists E' such that $E' \to \infty$ as $E \to \infty$ and

$$\pi(F_p(\beta, \boldsymbol{\mu}(\beta))) + \sigma_p - P_{E'}(\beta, \boldsymbol{\mu}(\beta) \mid p) = o(e^{-\beta E}) \qquad \text{as} \quad \beta \to \infty$$

for any curve $\boldsymbol{\mu}(\beta)$ along which G_p is stable. Using (5.1), we thus obtain

$$P^{\text{th}}(\beta, \boldsymbol{\mu}(\beta)) + \mathbf{e}_p \cdot \boldsymbol{\mu}(\beta) - P_{E'}(\beta, \boldsymbol{\mu}(\beta) \mid p) = o(e^{-\beta E}) \tag{5.3}$$

If $\dot{\boldsymbol{\mu}}(\beta)$ is the asymptotic series of $\boldsymbol{\mu}(\beta)$, then (5.3) implies that $\dot{P}(\beta, \dot{\boldsymbol{\mu}}(\beta) \mid p)$ is the asymptotic expansion of $P^{\text{th}}(\beta, \boldsymbol{\mu}(\beta)) + \mathbf{e}_p \cdot \boldsymbol{\mu}(\beta)$, which proves the Theorem 1.

In order to prove Theorem 2, let $\boldsymbol{\mu}(\beta)$ be the curve for which all phases coexist. We then have from (5.3)

$$(\mathbf{e}_p - \mathbf{e}_1) \cdot \boldsymbol{\mu}(\beta) = P_{E'}(\beta, \boldsymbol{\mu}(\beta) \mid p) - P_{E'}(\beta, \boldsymbol{\mu}(\beta) \mid 1) + o(e^{-\beta E}), \qquad p = 2, \ldots, d \tag{5.4}$$

i.e.,

$$\boldsymbol{\mu}(\beta) = \mathbf{f}_{E'}(\boldsymbol{\mu}(\beta)) + o(e^{-\beta E}) \tag{5.5}$$

where

$$\mathbf{f}_{E'}(\boldsymbol{\mu}) = \mathbf{E}^{-1}\,\Delta P_{E'}(\beta, \boldsymbol{\mu}) \tag{5.6}$$

[cf. Eq. (4.2)]. Consider now

$$\boldsymbol{\mu}_N(\beta) = \sum_{n=0}^{N} \boldsymbol{\mu}_n e^{-\beta E_n}, \qquad E_N \leqslant E', E \tag{5.7}$$

where $\boldsymbol{\mu}_n$ are the coefficients of the solution of Eq. (4.2). One can see that

$$\boldsymbol{\mu}_N(\beta) = \mathbf{f}_{E'}(\boldsymbol{\mu}_N(\beta)) + o(e^{-\beta E_N}) \tag{5.8}$$

From Eqs. (5.5) and (5.8), we want to conclude that

$$\boldsymbol{\mu}(\beta) - \boldsymbol{\mu}_N(\beta) = o(e^{-\beta E_N}) \qquad \text{for all } N \tag{5.9}$$

i.e., that the asymptotic expansion fo $\boldsymbol{\mu}(\beta)$ exists. Under this assumption, we already proved in the previous section that $\dot{\boldsymbol{\mu}}(\beta)$ is given by Eq. (4.3).

Equation (5.9) results from the following observation. The term of zero order of $P_E(\beta, \boldsymbol{\mu} \mid p)$ is independent of $\boldsymbol{\mu}$. One can therefore see that for any $\boldsymbol{\mu}, \mathbf{v} \in \mathbb{R}^{d-1}$,

$$|\mathbf{f}_{E'}(\boldsymbol{\mu}) - \mathbf{f}_{E'}(\mathbf{v})| \leqslant C_1(\boldsymbol{\mu}, \mathbf{v}) e^{-\beta E_1} |\boldsymbol{\mu} - \mathbf{v}| \tag{5.10}$$

where $C_1(\boldsymbol{\mu}, \mathbf{v})$ is continuous. Replacing (5.5) and (5.8) by inequalities and taking their differences, we obtain

$$|\boldsymbol{\mu}(\beta) - \boldsymbol{\mu}_N(\beta)| \leqslant |\mathbf{f}_{E'}(\boldsymbol{\mu}(\beta)) - \mathbf{f}_{E'}(\boldsymbol{\mu}_N(\beta))| + C_2(\beta, \boldsymbol{\mu}(\beta), \boldsymbol{\mu}_N(\beta)) e^{-\beta E_N} \tag{5.11}$$

where $C_2 \to 0$ as $\beta \to \infty$. Let us insert (5.10) into (5.11); then

$$|\boldsymbol{\mu}(\beta) - \boldsymbol{\mu}_N(\beta)| \leqslant C_1 e^{-\beta E_1} |\boldsymbol{\mu}(\beta) - \boldsymbol{\mu}_N(\beta)| + C_2 e^{-\beta E_N} \tag{5.12}$$

Iterating this inequality n times with n chosen so that $nE_1 > E_N$, we obtain

$$|\boldsymbol{\mu}(\beta) - \boldsymbol{\mu}_N(\beta)| \leqslant C_2 e^{-\beta E_N} \sum_{k=0}^{n-1} C_1^k e^{-\beta k E_1} + C_1^n e^{-\beta n E_1} |\boldsymbol{\mu}(\beta) - \boldsymbol{\mu}_N(\beta)| \tag{5.13}$$

Noticing that $\boldsymbol{\mu}(\beta)$ and $\boldsymbol{\mu}_N(\beta)$ are bounded in a neighborhood of $\beta = \infty$, Eq. (5.9) is established.

6. PHASE DIAGRAM TO THE LOWEST ORDER

As is well known from the PS theory, Eq. (5.1) generates the phase diagram $\{\Lambda_I(T)\}$ at positive temperatures T. It is understood that the construction is valid in a neighborhood of $\boldsymbol{\lambda}_0 = \mathbf{0}$. Primarily, one obtains a partition $\{M_I(T)\}$ of the space $\mathbb{R}^{d-1}$ of vectors $\boldsymbol{\mu}$, defined by

$$M_I(T) = \{\boldsymbol{\mu} \in \mathbb{R}^{d-1} \mid \pi(F_p(1/T, \boldsymbol{\mu})) + \varphi_p(\boldsymbol{\mu}) \text{ is maximal for } p \in I\} \quad (6.1)$$

for any nonempty $I \subset \{1, 2,..., d\}$ and

$$\varphi_p(\boldsymbol{\mu}) = \sigma_p - \mathbf{e}_p \cdot \boldsymbol{\mu} \quad (6.2)$$

Then, $M_I(T)$ is mapped onto the parameter space by $\boldsymbol{\mu} \to \boldsymbol{\lambda} = T\boldsymbol{\mu}$ [cf. Eq. (2.12)], giving

$$\Lambda_I(T) = TM_I(T) = \{T\boldsymbol{\mu} \mid \boldsymbol{\mu} \in M_I(T)\} \quad (6.3)$$

For $|I| \geqslant 2$, we can use the asymptotic expansion of $M_I(T)$ (cf. Theorem 2) to write

$$\Lambda_I(T) = \{T(\boldsymbol{\mu} + \mathbf{r}_T(\boldsymbol{\mu})) \mid \boldsymbol{\mu} \in M_I(0)\} \approx TM_I(0)\} \quad (6.4)$$

where $\mathbf{r}_T(\boldsymbol{\mu})$ is a vector field on $M_I(0)$ (nonunique) and

$$|\mathbf{r}_T(\boldsymbol{\mu})| \leqslant C \exp(-E_1/T) \qquad \text{as} \quad T \to 0$$

uniformly in a neighborhood of $\boldsymbol{\mu}_0$ [see Eq. (4.6)].

It turns out that this lowest order approximation is nontrivial if σ_p is not the same for all p. Since $\pi(F_p(\beta, \boldsymbol{\mu})) \to 0$ as $\beta \to \infty$, we obtain from Eqs. (5.1) and (6.1), respectively,

$$\lim_{\beta \to \infty} P^{\text{th}}(\beta, \boldsymbol{\mu}) = \max_p \{\varphi_p(\boldsymbol{\mu})\} \quad (6.5)$$

and

$$M_I(0) = \{\boldsymbol{\mu} \in \mathbb{R}^{d-1} \mid \varphi_p(\boldsymbol{\mu}) \text{ is maximal for } p \in I\} \quad (6.6)$$

By comparison with

$$\Lambda_I(0) = \{\boldsymbol{\lambda} \in \mathbb{R}^{d-1} \mid \mathbf{e}_p \cdot \boldsymbol{\lambda} \text{ is minimal for } p \in I\} \quad (6.7)$$

and recalling that $\sigma_p - \mathbf{e}_p \cdot \boldsymbol{\mu}_0$ is independent of p [cf. Eqs. (4.1) and (4.6)], we see that

$$TM_I(0) = \Lambda_I(0) + T\boldsymbol{\mu}_0 \quad (6.8)$$

and therefore

$$\Lambda_I(T) \approx \Lambda_I(0) + T\boldsymbol{\mu}_0 \tag{6.9}$$

If σ_p varies with p, then $\boldsymbol{\mu}_0 \neq 0$ and we get an overall shift as a first correction to the zero-temperature phase diagram.

Let $\boldsymbol{\mu} \in M_p(0)$; then $\varphi_p(\boldsymbol{\mu}) > \varphi_q(\boldsymbol{\mu})$ for all $q \neq p$, and there exists a $T_0 = T_0(\boldsymbol{\mu})$ such that $\boldsymbol{\mu} \in M_p(T)$ if $T < T_0$ and hence $T\boldsymbol{\mu} \in \Lambda_p(T)$. Thus, for $T < T_0$ there exists a pure phase of $H_0 + T\boldsymbol{\mu} \cdot \mathbf{H}'$ associated with G_p and there is none associated with G_q if $q \neq p$.

For $\boldsymbol{\mu} = 0$, $\varphi_q(\boldsymbol{\mu}) = \sigma_q$, $q = 1,..., d$; therefore, the pure phases of H_0 near $T = 0$ belong to the classes with largest residual entropy, a result which was earlier obtained by Aizenman and Lieb.[9] This is in agreement with the principle of domination,[4] when domination takes places at the lowest order.

Equation (6.5) tells us that $P^{\text{th}}(\infty, \boldsymbol{\mu})$ is the convex envelope of the affine functions $\varphi_1(\boldsymbol{\mu}),..., \varphi_d(\boldsymbol{\mu})$. In Fig. 1, we have fixed a unit vector $\mathbf{a} \in \mathbb{R}^{d-1}$ and plotted $\varphi_p(\mu\mathbf{a})$ versus μ for all p. For generic $\mathbf{a}$ the linear parts of the convex envelope belong to a unique class G_p and for those values of μ there exists a pure phase corresponding to G_p at low temperatures. The vertices give the slope of the coexistence curves at $T = 0$.

One often represents $\{\Lambda_I(T)\}$ in the space $(\boldsymbol{\lambda}, T)$. The counterpart of Fig. 1 is obtained if we cut the extended phase diagram $\{\Lambda_I(T), T)\}$ by the plane $(\boldsymbol{\lambda} = \lambda\mathbf{a}, T)$. This yields the phase diagram corresponding to the Hamiltonian $H_0 + \lambda\mathbf{a} \cdot \mathbf{H}'$. The result is shown in the approximation (6.9), in Fig. 2. The angular domain i_k corresponds to $\lambda = T\mu$, with μ varying in the interval where $P^{\text{th}}(\infty, \boldsymbol{\mu}) = \varphi_{i_k}$, and the broken line to $\lambda = T\mu_{0,i}$, where $\mu_{0,i}$ is the abscissa of a vertex of the convex envelope. Let us notice that fixing $\boldsymbol{\mu} = \mu\mathbf{a}$ corresponds to a situation when the degeneracy of $d > 2$

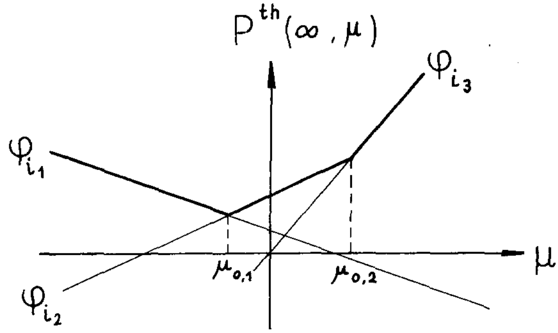

Fig. 1. $\varphi_p = \sigma_p - \mu\mathbf{e}_p \cdot \mathbf{a}$ versus μ for three phases. The thermodynamic pressure at $T = 0$ is their convex envelope [cf. Eq. (6.5)].

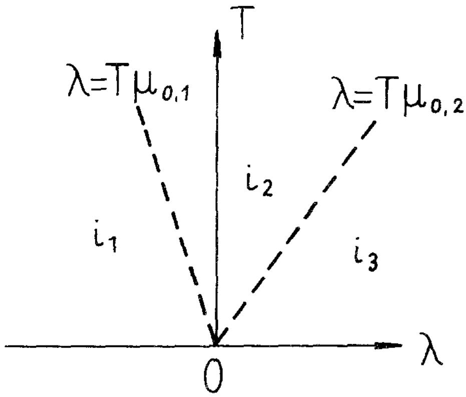

Fig. 2. Phase diagram corresponding to Fig. 1, in the lowest order approximation.

classes is partially lifted by a single "external field" H' $(= \mathbf{a} \cdot \mathbf{H}')$. The analogous problem for systems without residual entropy was studied by Tarnawski.[11]

7. EXAMPLES

7.1. Griffiths–Bernasconi–Rys Model[12,13]

7.1.1. Definition of the Model. The GBR model is a spin-1 model on $\mathbb{Z}^2$ defined by the formal Hamiltonian

$$H = -K \sum_{nn} x_i^2 x_j^2 - g \sum_i x_i^2 - h \sum_i x_i, \qquad \Omega_0 = \{-1, 0, 1\} \tag{7.1}$$

where K is a positive constant and g, h are real parameters. For fixed h at low temperatures there is a phase transition as g varies.

This model is interesting for two reasons. First, for $h \neq 0$ the phase transition is between two phases associated with two gsc and Slawny's method applies; but for $h = 0$, the phase transition takes place between one phase associated with a gsc and another phase associated with a class of gsc with nonvanishing entropy. The second reason is that this model is equivalent to the 2D Ising model and the coexistence surface can be computed explicitly: it is given by the equation

$$2K + g + |h| + \frac{1}{\beta} \ln(1 + e^{-2\beta|h|}) = 0 \tag{7.2}$$

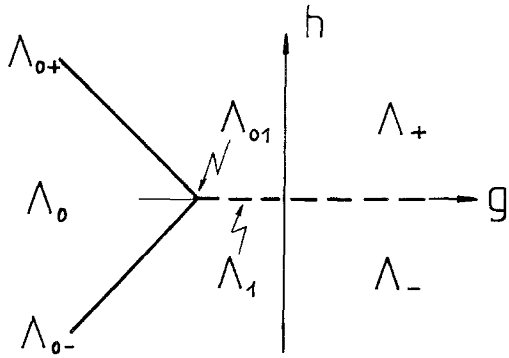

Fig. 3. Zero-temperature phase diagram for the GBR model.

Therefore it will be possible to compare the asymptotic expansion of the coexistence surface with the exact result and to understand in this case the mechanism by which the entropy contribution appears in the limit $h \to 0$.

The phase diagrams are shown in Figs. 3 and 4. The four classes we will have to consider are

$$G_0 = \{x^0 \equiv 0\}; \qquad G_+ = \{x^+ \equiv 1\}$$
$$G_- = \{x^- \equiv -1\}; \qquad G_1 = \{x \mid x_i^2 = 1 \text{ for all } i\} \tag{7.3}$$

with residual entropy

$$\sigma_0 = \sigma_+ = \sigma_- = 0, \qquad \sigma_1 = \ln 2$$

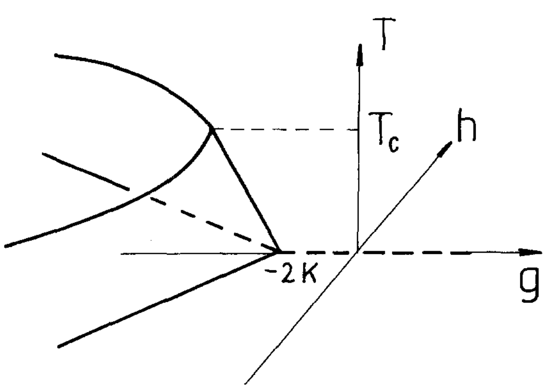

Fig. 4. Phase diagram for the GBR model at $T > 0$.

For a given value of h we consider the unperturbed Hamiltonian H_0 corresponding to the point on the coexistence line, i.e., $g = -2K - |h|$ and

$$H_0 = \frac{K}{2} \sum_{nn} (x_i^2 - x_j^2)^2 + \sum_i x_i(|h| x_i - h) \tag{7.4}$$

$$\begin{aligned} &\text{if} \quad h > 0, \quad \text{then} \quad G[H_0] = G_0 \cup G_+ \\ &\text{if} \quad h < 0, \quad \text{then} \quad G[H_0] = G_0 \cup G_- \\ &\text{if} \quad h = 0, \quad \text{then} \quad G[H_0] = G_0 \cup G_1 \end{aligned}$$

(For $h = 0$ and $g \neq -2K$, there is one gsc x^0 if $g < -2K$, and one class G_1 if $g > -2K$.)

For each value of h, $G[H_0]$ consists of two classes and thus $r = d - 1 = 1$. The perturbation

$$H' = -\sum_i x_i^2 \tag{7.5}$$

splits the degeneracy, and the one-parameter family of Hamiltonians which we consider is

$$H = H_0 + \lambda H' \tag{7.6}$$

It corresponds to the original Hamiltonian (7.1) with

$$g = \lambda - 2K - |h| = \frac{\mu}{\beta} - 2K - |h| \tag{7.7}$$

For this model the cells A_α are simply the sites of $\mathbb{Z}^2$.

7.1.2. Case $h = 0$: $G[H_0] = G_0 \cup G_1$. For $h = 0$ we have $E_n = (n+1)K$ $(n \geqslant 1)$, $e_0 = 0$, $e_1 = -1$; $H'(\xi) = -|S(\xi)|$ for $p = 0$, and $H'(\xi) = |S(\xi)|$ for $p = 1$ boundary conditions. Since

$$\varphi_0 = \sigma_0 - e_0 \mu = 0$$

$$\varphi_1 = \sigma_1 - e_1 \mu = \ln 2 + \mu$$

then

$$\mu_0 = \mu(T = 0) = \frac{\sigma_0 - \sigma_1}{e_0 - e_1} = -\ln 2 \qquad \text{[see Eq. (4.6)]}$$

and we obtain the phase diagram to lowest order (Fig. 5): for $\mu < -\ln 2$ $(g < -2K - T \ln 2)$, the phase G_0 is stable, while for $\mu > -\ln 2$

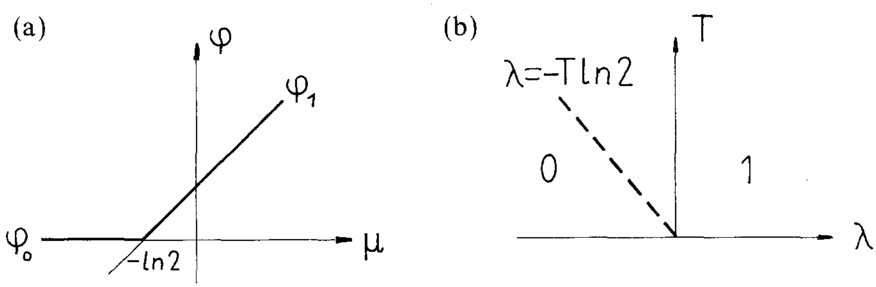

Fig. 5. Phase diagram of the GBR model in the lowest order for $h=0$ [cf. Eq. (7.7)]. It coincides with the exact solution.

$(g > -2K - T\ln 2)$, the phase G_1 is stable. In particular, for H_0 $(g = -2K)$, the stable phase is G_1.

Computing the first coefficients $a_n(\mu \mid p)$ of the expansion (3.19), we find

$$\dot{P}(\beta, \mu \mid 0) = z e^{-2\beta K} + 2z^2 e^{-3\beta K} + (z^4 + 6z^3 - \tfrac{5}{2}z^2) e^{-4\beta K} + \cdots$$
$$\dot{P}(\beta, \mu \mid 1) = \ln 2 + z^{-1} e^{-2\beta K} + 2z^{-2} e^{-3\beta K} + (z^{-4} + 6z^{-3} - \tfrac{5}{2}z^{-2}) e^{-4\beta K} + \cdots \tag{7.8}$$

where $z = 2e^{\mu}$.

The fixed-point equation (4.2) is therefore

$$\ln \dot{z} - \ln 2 = -\ln 2 + \sum_{n \geq 2} [P_m(\dot{z}) - P_m(\dot{z}^{-1})] e^{-n\beta K} \tag{7.9}$$

where P_m is a polynomial of order m, $m = m(n)$, with $m = O(n^2)$.

The solution of (7.9) is $\dot{z} = 1$, i.e.,

$$\dot{\mu}(\beta) = -\ln 2 \qquad \text{or} \qquad \lambda(T) = -T\ln 2$$

We have thus obtained the coexistence curve of (7.1) for $h=0$

$$g = -2K - T\ln 2$$

which is the exact solution (7.2).

For $\mu \leq -\ln 2$ $(z \leq 1)$ G_0 is stable and the asymptotic expansion of the thermodynamic pressure is given by $\dot{P}(\beta, \mu \mid 0)$; for $\mu \geq -\ln 2$ $(z \geq 1)$, G_1 is stable and the asymptotic expansion is given by $\mu + \dot{P}(\beta, \mu \mid 1)$.

7.1.3. Case $h>0$: $G[H_0]=G_0 \cup G+$. For $h>0$ we have $E_n=pK+qh$ (p, q integers), $e_0=0$, $e_+=-1$; $H'(\xi)=-|S(\xi)|$ for 0 and $H'([x])=|\{i \mid x_i=0\}|$ for + boundary conditions. Since

$$\varphi_0=\sigma_0-e_0\mu=0$$

$$\varphi_+=\sigma_+-e_+\mu=\mu$$

then

$$\mu_0=\mu(T=0)=\frac{\sigma_0-\sigma_+}{e_0-e_+}=0$$

and we obtain the phase diagram to lowest order (Fig. 6): for $\mu<0$ ($g<-2K-h$) the phase G_0 is table, while for $\mu>0$ ($g>-2K-h$) the phase G_+ is stable. From the zeroth order we cannot decide which of G_0 and G_1 is stable for $g=-2K-h$.

Computing the first coefficients $a_n(\mu \mid p)$, we obtain

$$\dot{P}(\beta, \mu \mid 0)=\xi e^{-2\beta K}+2\xi^2 e^{-3\beta K}+(\xi^4+6\xi^3-\tfrac{5}{2}\xi^2)e^{-4\beta K}+\cdots$$

$$\dot{P}(\beta, \mu \mid +)=\ln(1+e^{-2\beta K})+\xi^{-1}e^{-2\beta K}+2\xi^{-2}e^{-3\beta K}$$
$$+(\xi^{-4}+6\xi^{-3}-\tfrac{5}{2}\xi^{-2})e^{-4\beta K}+\cdots \qquad (7.10)$$

where

$$\xi=e^{\mu}(1+e^{-2\beta h})$$

The fixed-point equation (4.2) is

$$\ln \dot{\xi}-\ln(1+e^{-2\beta h})=-\ln(1+e^{-2\beta h})+\sum_{n\geq 2}[P_m(\dot{\xi})-P_m(\dot{\xi}^{-1})]e^{-n\beta K}$$

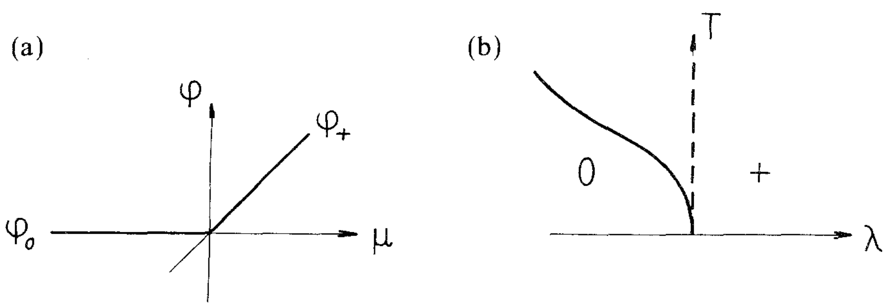

Fig. 6. Phase diagram of the GBR model in the lowest order for $h\neq 0$ (dashed line), and the exact solution (full line).

which gives $\dot{\xi}=1$, i.e.,

$$\dot{\mu}(\beta)=\sum_{n\geq 1}\frac{(-1)^n}{n}e^{-n2\beta h}=-\ln(1+e^{-2\beta h})=\mu(\beta)$$

We have thus obtained the coexistence curve of (7.1) for $h>0$:

$$g=-2K-h-T\ln(1+e^{-2\beta h})$$

For $\mu<0$, G_0 is stable and the asymptotic expansion of thermodynamic pressure is given by $\dot{P}(\beta,\mu\,|\,0)$, while for $\mu\geq 0$, G_+ is stable and the asymptotic expansion is given by $\mu+\dot{P}(\beta,\mu\,|\,+)$. In particular, for H_0 $(g=-2K-h)$ the stable phase is G_+.

To conclude this discussion, we note that one passes continuously from the phase G_+ to the phase G_-: there is no phase transition at $h=0$ for $\mu_c>-2K$ (Fig. 4).

7.2. Magnetic Lattice Gas Model I

7.2.1. Definition of the Model. This model, which we have previously discussed,[1] is a spin-1 model on $\mathbb{Z}^2$ defined by

$$H=-\sum_{\mathrm{nn}}(Jx_ix_j+K_1x_i^2x_j^2)-K_2\sum_{\mathrm{nnn}}x_i^2x_j^2-g\sum_i x_i^2 \tag{7.11}$$

We consider the case where J and K_1 are constant, $J>0$, $J+K_1<0$; K_2 and g are real parameters with $K_2>0$.

The zero-temperature phase diagram was described in ref. 1 and is reproduced in Fig. 7.

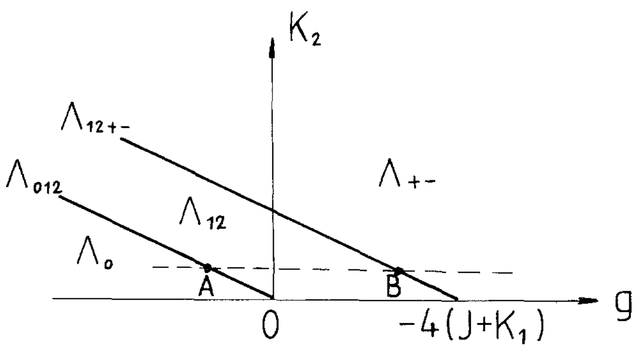

Fig. 7. Zero temperature phase diagram for the magnetic lattice gas model [Eq. (7.11)].

The five classes we have to consider are the following:

$$G_0 = \{x^0 \equiv 0\}; \qquad G_+ = \{x^+ \equiv 1\}; \qquad G_- = \{x^- \equiv -1\}$$
$$G_1 = \{x \mid x_i = 0 \text{ for } i_1 + i_2 \text{ even}, x_i^2 = 1 \text{ for } i_1 + i_2 \text{ odd}\}$$
$$G_2 = \{x \mid x_i^2 = 1 \text{ for } i_1 + i_2 \text{ even}, x_i = 0 \text{ for } i_1 + i_2 \text{ odd}\}$$

with residual entropy

$$\sigma_0 = \sigma_+ = \sigma_- = 0, \qquad \sigma_1 = \sigma_2 = \tfrac{1}{2}\ln 2$$

The two classes (G_+, G_-) and the two classes (G_1, G_2) are related by symmetry.

For a given value of K_2, we consider the unperturbed Hamiltonian corresponding to the points on the coexistence lines, i.e.,

(A) $g = -2K_2$, $\qquad G[H_0] = G_0 \cup G_1 \cup G_2$

(B) $g = -2K_2 - 4(J + K_1)$, $\qquad G[H_0] = G_1 \cup G_2 \cup G_+ \cup G_-$

7.2.2. Coexistence of the Phases G_0, G_1, G_2. The unperturbed Hamiltonian is

$$H_0 = -\sum_{\text{nn}} (Jx_i x_j + K_1 x_i^2 x_j^2) + \tfrac{1}{2}K_2 \sum_{\text{nnn}} (x_i^2 - x_j^2)^2 \tag{7.12}$$

and we consider the one-parameter family of Hamiltonians

$$H = H_0 + \lambda H', \qquad H' = -\sum_i x_i^2$$

(which does not split the degeneracy associated with symmetry).

The connection with the original Hamiltonian (7.11) is given by

$$g = \lambda - 2K_2 = \frac{\mu}{\beta} - 2K_2 \tag{7.13}$$

(The cells A_α are plaquettes with four sites.) Since

$$e_0 = 0, \qquad e_1 = e_2 = -\tfrac{1}{2}$$

we have

$$\varphi_0 = \sigma_0 - e_0\mu = 0$$
$$\varphi_1 = \varphi_2 = \sigma_1 - e_1\mu = \tfrac{1}{2}\ln 2 + \tfrac{1}{2}\mu$$

Thus

$$\mu_0 = \mu(T=0) = \frac{\sigma_0 - \sigma_1}{e_0 - e_1} = -\ln 2$$

which yields the phase diagram to lowest order (Fig. 8): the phase G_0 is stable for $\mu < -\ln 2$ $(g < -2K_2 - T\ln 2)$, and the phases G_1, G_2 are stable for $\mu > -\ln 2$ $(g > -2K_2 - T\ln 2)$. In particular, for H_0 $(g = -2K_2)$ the stable phases are G_1 and G_2.

Computing the first coefficients $a_n(\mu \mid p)$ with $K_2 > J + 2|J + K_1|$, we find

$$\dot{P}(\beta, \mu \mid 0) = 2e^{\mu}e^{-2\beta K} + 8e^{2\mu}e^{-3\beta K_2} + (48e^{3\mu} - 18e^{2\mu})e^{-4\beta K_2} + \cdots$$

$$\dot{P}(\beta, \mu \mid 1) = \tfrac{1}{2}\ln 2 + \tfrac{1}{4}e^{-\mu}e^{-2\beta K_2} + \tfrac{1}{16}e^{\mu}e^{-2\beta(K_2 - 2J - 2K_1)} + \tfrac{1}{4}e^{\mu}e^{-2\beta(K_2 - J - 2K_1)} + \cdots$$

The fixed-point equation (4.2)

$$\dot{\mu} = -\ln 2 + (4e^{\dot{\mu}} - \tfrac{1}{2}e^{-\dot{\mu}})e^{-2\beta K_2} - \tfrac{1}{8}e^{\dot{\mu}}e^{-2\beta(K_2 - 2J - 2K_1)} - \tfrac{1}{2}e^{\dot{\mu}}e^{-2\beta(K_2 - J - 2K_1)} + \cdots$$

gives the asymptotic expansion for the coexistence curve:

$$\mu(\beta) = -\ln 2 + e^{-2\beta K_2} - \tfrac{1}{16}e^{-2\beta(K_2 - 2J - 2K_1)} - \tfrac{1}{4}e^{-2\beta(K_2 - J - 2K_1)} + \cdots$$

Therefore the coexistence curve is

$$g(T) = -2K_2 + T[-\ln 2 + e^{-2\beta K_2} - \tfrac{1}{16}e^{-2\beta(K_2 - 2J - 2K_1)} + \cdots]$$

For $\mu \leqslant -\ln 2$, G_0 is stable and the asymptotic expansion for the thermodynamic pressure is given by $\dot{P}(\beta, \mu \mid 0)$, while for $\mu > -\ln 2$, G_1 is stable and the asymptotic expansion is given by $\tfrac{1}{2}\mu + \dot{P}(\beta, \mu \mid 1)$. The phase diagram is shown in Fig. 10.

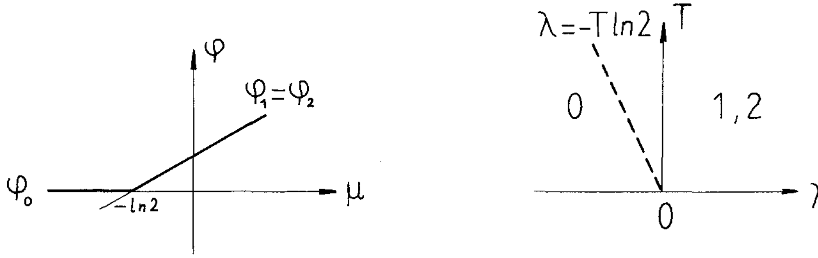

Fig. 8. Phase diagram of the magnetic lattice gas model in the lowest order around point A of Fig. 7.

7.2.3. Coexistence of the Phases G_1, G_2, $G+$, G_-. In this case, the unperturbed Hamiltonian and the one-parameter family of Hamiltonians are given by

$$H_0 = -\sum_{\mathrm{nn}} [Jx_ix_j + K_1x_i^2x_j^2 - (J+K_1)(x_i^2+x_j^2-1)] + \tfrac{1}{2}K_2 \sum_{\mathrm{nnn}} (x_i^2 - x_j^2)^2$$

$$H = H_0 + \lambda H', \qquad H' = -\sum_i x_i^2$$

and the connection with (7.11) is given by

$$g = \lambda - 2K_2 - 4(J+K_1)$$

Notice that we subtracted an infinite constant term in order that H_0 vanishes on gsc. Since

$$e_1 = e_2 = -\tfrac{1}{2}, \qquad e_+ = e_- = -1$$

we have

$$\varphi_1 = \varphi_2 = \sigma_1 - e_1\mu = \tfrac{1}{2}\ln 2 + \tfrac{1}{2}\mu$$

$$\varphi_+ = \varphi_- = \sigma_+ - e_+\mu = \mu$$

Thus

$$\mu_0 = \mu(T=0) = \frac{\sigma_1 - \sigma_+}{e_1 - e_+} = \ln 2$$

which gives the phase diagram to lowest order (Fig. 9): the phases G_1, G_2 are stable for $\mu < \ln 2$ ($g < -2K_2 + 4|J+K_1| + T\ln 2$), while the phases

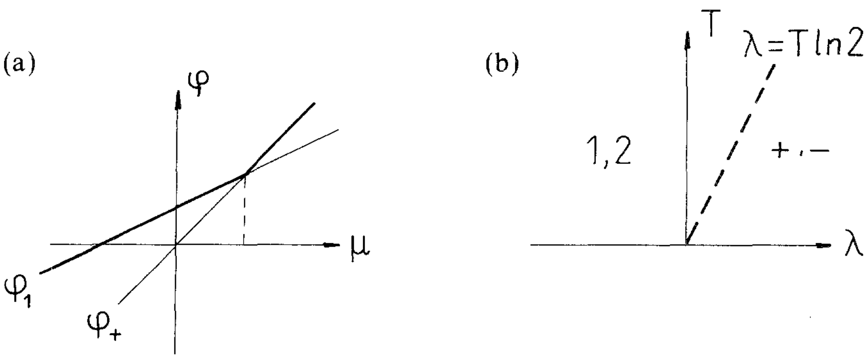

Fig. 9. Phase diagram of the magnetic lattice gas model in the lowest order around point B of Fig. 7.

G_+, G_- are stable for $\mu > \ln 2$ $(g > -2K_2 + 4|J+K_1| + T\ln 2)$. In particular, for H_0 $(g = 2K_2 + 4|J+K_1|)$ the stable phases are G_1 and G_2.

We obtain

$$\dot{P}(\beta, \mu \mid +) = e^{-\mu} e^{-2\beta K_2} + e^{-8\beta J} + 2e^{-2\mu} e^{-3\beta K_2} + \cdots$$

$$\dot{P}(\beta, \mu \mid 1) = \tfrac{1}{2} \ln 2 + \tfrac{1}{16} e^{\mu} e^{-2\beta K_2} + \tfrac{1}{4} e^{-\mu} e^{-2\beta(K_2 - 2J - 2K_1)} + \cdots$$

and the fixed-point equation (4.2) reads

$$\dot{\mu} = \ln 2 + (\tfrac{1}{8} e^{\dot{\mu}} - 2e^{-\dot{\mu}}) e^{-2\beta K_2} + \tfrac{1}{2} e^{-\dot{\mu}} e^{-2\beta(K_2 - 2J - 2K_1)} + \cdots$$

which implies

$$\mu(\beta) = \ln 2 - \tfrac{3}{4} e^{-2\beta K_2} + \tfrac{1}{4} e^{-2\beta(K_2 - 2J - 2K_1)} + \cdots$$

The corresponding coexistence curve is shown in Fig. 10:

$$g(T) = -2K_2 - 4(J + K_1) + T(\ln 2 - \tfrac{3}{4} e^{-2\beta K_2} + \cdots)$$

For $\mu < \ln 2$, G_1 and G_2 are stable and the asymptotic expansion of the thermodynamic pressure is given by $\frac{1}{2}\mu + \dot{P}(\beta, \mu \mid 1)$, while for $\mu \geqslant \ln 2$, G_+ and G_- are stable and the asymptotic expansion is given by $\mu + \dot{P}(\beta, \mu \mid +)$.

7.2.4. Magnetic Lattice Gas Model II. We consider the unperturbed Hamiltonian (7.12) with $J + K_1 = 0$ $(K = K_2)$:

$$H_0 = -J \sum_{\text{nn}} x_i x_j (1 - x_i x_j) + \tfrac{1}{2} K \sum_{\text{nnn}} (x_i^2 - x_j^2)^2 \qquad (7.14)$$

$$G[H_0] = G_0 \cup G_1 \cup G_2 \cup G_+ \cup G_-$$

$$\sigma_0 = \sigma_+ = \sigma_- = 0, \qquad \sigma_1 = \sigma_2 = \tfrac{1}{2} \ln 2$$

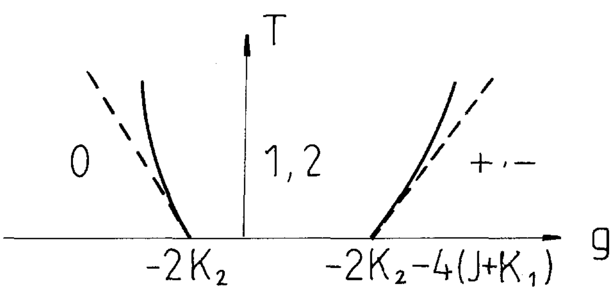

Fig. 10. Coexistence curve for the magnetic lattice gas model.

To study the phase diagram, we introduce the two-parameter family of Hamiltonians (which do not lift the degeneracy associated with symmetry):

$$H = H_0 + \lambda^{(1)} H^{(1)} + \lambda^{(2)} H^{(2)}$$

$$H^{(1)} = -\sum_{\text{nn}} x_i x_j, \qquad H^{(2)} = -\sum_{\text{nnn}} x_i^2 x_j^2$$

The zero-temperature phase diagram is shown in Fig. 11. Since

$$\mathbf{e}_0 = (0, 0), \qquad \mathbf{e}_1 = \mathbf{e}_2 = (0, -1), \qquad \mathbf{e}_+ = \mathbf{e}_- = (-2, -2)$$

we have

$$\varphi_0 = \sigma_0 - \mathbf{e}_0 \cdot \boldsymbol{\mu} = 0$$

$$\varphi_1 = \varphi_2 = \sigma_1 - \mathbf{e}_1 \cdot \boldsymbol{\mu} = \tfrac{1}{2} \ln 2 + \mu^{(2)}$$

$$\varphi_+ = \varphi_- = \sigma_+ - \mathbf{e}_+ \cdot \boldsymbol{\mu} = 2\mu^{(1)} + 2\mu^{(2)}$$

The phase diagram to lowest order is given by

$$M_0(0) = \{\boldsymbol{\mu} \mid \tfrac{1}{2} \ln 2 + \mu^{(2)} < 0,\ \mu^{(1)} + \mu^{(2)} < 0\}$$

$$M_{1,2}(0) = \{\boldsymbol{\mu} \mid \tfrac{1}{2} \ln 2 + \mu^{(2)} > 0,\ \tfrac{1}{2} \ln 2 - \mu^{(2)} - 2\mu^{(1)} > 0\}$$

$$M_{1+-}(0) = \{\boldsymbol{\mu} \mid \mu^{(1)} + \mu^{(2)} > 0,\ \tfrac{1}{2} \ln 2 - \mu^{(2)} - 2\mu^{(1)} < 0\}$$

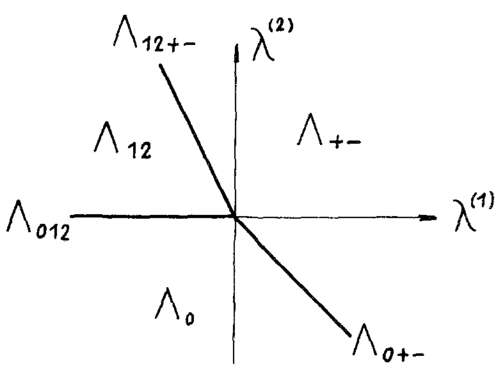

Fig. 11. Zero temperature phase diagram for the model (7.14).

which implies $\boldsymbol{\mu}_0=\boldsymbol{\mu}(T=0)=(\frac{1}{2}\ln 2,\ -\frac{1}{2}\ln 2)$; these are represented in Fig. 12. We then obtain the asymptotic expansion

$$\dot{P}^{\text{th}}(\beta, \boldsymbol{\mu}\,|\,0)=2e^{-2\beta K}+8e^{\mu^{(2)}}e^{-3\beta K}+2(2e^{\mu^{(1)}}+24e^{2\mu^{(2)}}-9)e^{-4\beta K}+\cdots$$

$$\dot{P}^{\text{th}}(\beta, \boldsymbol{\mu}\,|\,+)=2(\mu^{(1)}+\mu^{(2)})+e^{-4(\mu^{(1)}+\mu^{(2)})}e^{-2\beta K}+e^{-8\mu^{(1)}}e^{-8\beta J}$$
$$+2e^{-(8\mu^{(1)}+6\mu^{(2)})}e^{-3\beta K}+\cdots$$

$$\dot{P}^{\text{th}}(\beta, \boldsymbol{\mu}\,|\,1)=\mu^{(2)}+\tfrac{1}{2}\ln 2+\tfrac{1}{4}(e^{-4\mu^{(2)}}+\tfrac{1}{4}e^{4\mu^{(1)}})e^{-2\beta K}$$
$$+\tfrac{1}{4}e^{2\mu^{(1)}}e^{-2\beta(K+J)}+\cdots$$

The fixed-point equations for the coexistence curve of all phases are

$$-\dot{\mu}^{(2)}=\tfrac{1}{2}\ln 2+\tfrac{1}{4}(e^{-4\dot{\mu}^{(2)}}+\tfrac{1}{4}e^{4\mu^{(1)}}-8)e^{-2\beta K}+\cdots$$
$$-2(\dot{\mu}^{(1)}+\dot{\mu}^{(2)})=(e^{-4(\dot{\mu}^{(1)}+\dot{\mu}^{(2)})}-2)e^{-2\beta K}+\cdots$$

which yields

$$\mu^{(1)}=\tfrac{1}{2}\ln 2-\tfrac{1}{4}e^{-2\beta K}+\cdots$$
$$\mu^{(2)}=-\tfrac{1}{2}\ln 2+\tfrac{3}{4}e^{-2\beta K}+\cdots$$

For the coexistence surface between the phases associated with G_0, G_+, G_-, we find

$$-2(\dot{\mu}^{(1)}+\dot{\mu}^{(2)})=(e^{-4(\dot{\mu}^{(1)}+\dot{\mu}^{(2)})}-2)e^{-2\beta K}+\cdots$$

Therefore

$$\mu^{(1)}+\mu^{(2)}=\tfrac{1}{2}e^{-2\beta K}+\cdots$$
$$\lambda^{(2)}=-\lambda^{(1)}+T[\tfrac{1}{2}e^{-2\beta K}+\cdots]$$

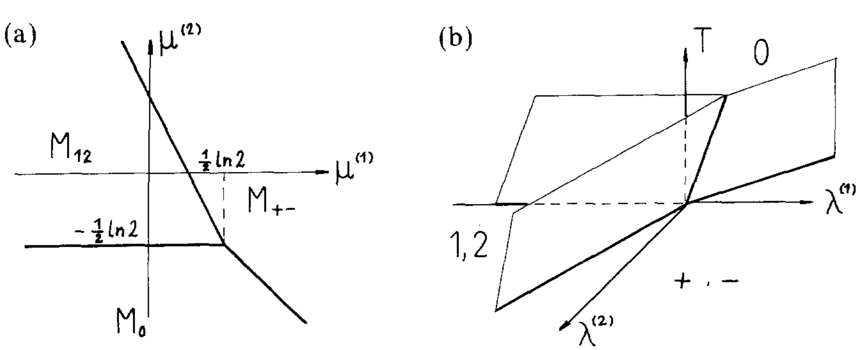

Fig. 12. Zero-order phase diagram for the model (7.14).

Similarly, for the coexistence surface between the phases G_0, G_1, G_2

$$-\dot{\mu}^{(2)} = \tfrac{1}{2}\ln 2 + \left(\tfrac{1}{4}e^{-4\dot{\mu}^{(2)}} + \tfrac{1}{16}e^{4\dot{\mu}^{(1)}} - 2\right)e^{-2\beta K} + \cdots$$

which gives

$$\lambda^{(2)} = T\left\{-\tfrac{1}{2}\ln 2 + \left(1 - \tfrac{1}{16}e^{4\beta\lambda^{(1)}}\right)e^{-2\beta K} + \cdots\right\}$$

Finally, for the coexistence surface between the phases G_1, G_2, G_+, G_-

$$2\dot{\mu}^{(1)} + \dot{\mu}^{(2)} = \tfrac{1}{2}\ln 2 + \left(\tfrac{1}{4}e^{-4\dot{\mu}^{(2)}} + \tfrac{1}{16}e^{4\dot{\mu}^{(1)}} - e^{-4(\dot{\mu}^{(1)} + \dot{\mu}^{(2)})}\right)e^{-2\beta K} + \cdots$$

gives

$$\lambda^{(2)} = -2\lambda^{(1)} + T\left\{\tfrac{1}{2}\ln 2 + \tfrac{1}{16}\left(e^{8\beta\lambda^{(1)}} - 3e^{4\beta\lambda^{(1)}}\right)e^{-2\beta K} + \cdots\right\}$$

These results are summarized in Fig. 13.

To conclude this example, we consider the case where the degeneracy of H_0, (7.14), is only partially lifted, i.e.,

$$H = H_0 + \lambda H^{(2)}$$

It corresponds to the original Hamiltonian (7.11) with $J + K_1 = 0$, $g = -2K$, $K_2 = K + \lambda$. The zero-temperature phase diagram is represented by Fig. 7, where the two lines coincide and correspond to the coexistence of the five classes.

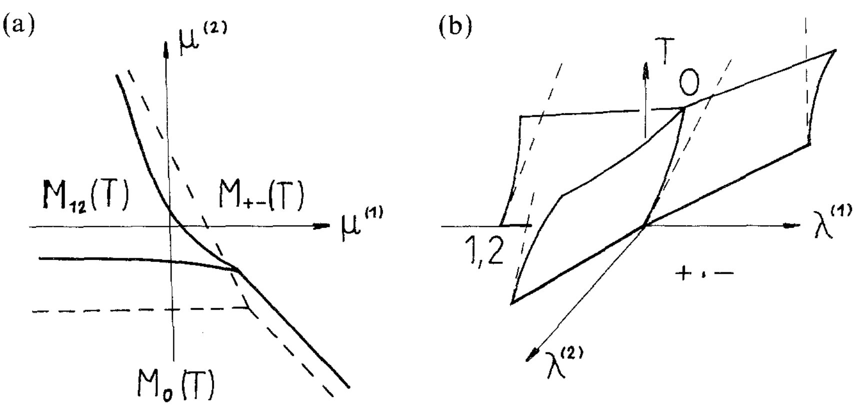

Fig. 13. First order phase diagram for the model (7.14).

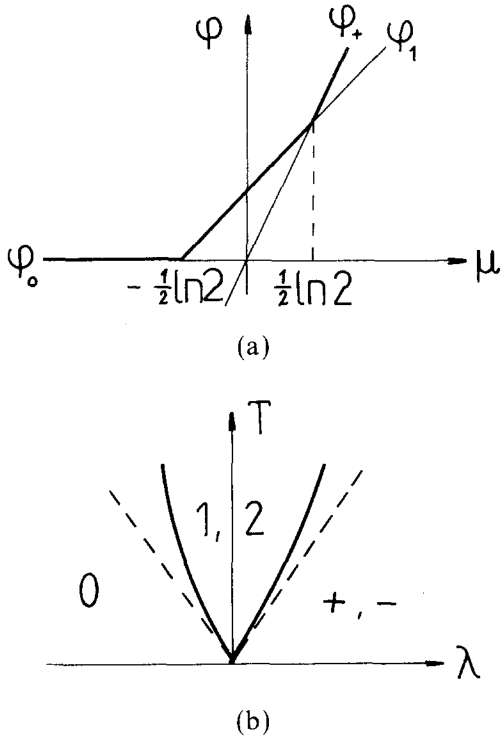

Fig. 14. Lowest order phase diagram for the model (7.11) $K_1 = -J$, $g = -2K$ and $K_2 = K + \lambda$.

The phase diagram to lowest order (Fig. 14) is obtained from

$$\varphi_0 = \sigma_0 - e_0 \mu = 0$$

$$\varphi_1 = \varphi_2 = \sigma_1 - e_1 \mu = \tfrac{1}{2} \ln 2 + \mu$$

$$\varphi_+ = \varphi_- = \sigma_+ - e_+ \mu = 2\mu$$

For $\mu < -\frac{1}{2} \ln 2$, the phase G_0 is stable; for $|\mu| < \frac{1}{2} \ln 2$, the phases G_1, G_2 are stable, and for $\mu > > \frac{1}{2} \ln 2$, the phases G_+, G_- are stable. In particular, for H_0 the phases G_1 and G_2 are stable. The asymptotic expansions for the coexistence curve for the phases G_0, G_1, G_2 is

$$\mu = -\tfrac{1}{2} \ln 2 + \tfrac{15}{16} e^{-2\beta K} + \cdots$$

and for the coexistence curve for the phases G_1, G_2, G_+, G_- it is

$$\mu = \tfrac{1}{2} \ln 2 + \tfrac{1}{8} e^{-2\beta K} + \cdots$$

APPENDIX

The combinatorial factor $C_A(\theta)$ is determined in the following way. For any $\theta \in [X^P]$ we construct a graph $g_A(\theta)$: (i) Any ξ such that $\theta(\xi) > 0$ is represented by a complete subgraph $g_\theta(\xi)$ of $\theta(\xi)$ vertices. Thus, $g_A(\theta)$ has altogether $\sum \theta(\xi)$ vertices. (ii) If $S(\xi_1) \bmod A$ and $S(\xi_2) \bmod A$ are connected, then $g_\theta(\xi_1) \cup g_\theta(\xi_2)$ is completed with the missing edges.

Now

$$C_A(\theta) = \begin{cases} 0 & \text{if for some } \xi \text{ s.t. } \theta(\xi) > 0, \\ & S(\xi) \text{ is not faithfully represented on } T_A \\ \sum\limits_{\gamma \subset g_A(\theta)} (-1)^{\#\{\text{edges of } \gamma\}} & \text{otherwise} \end{cases} \tag{A.1}$$

where the summation goes over connected subgraphs covering all the vertices of $g_A(\theta)$. If A is sufficiently large, then $g_A(\theta) = g(\theta)$ and $C_A(\theta) = C(\theta)$, i.e., they are independent of A. Notice that $C_A(\theta) = 0$ if $g_A(\theta)$ is disconnected.

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